

# The time of arrival concept in quantum mechanics<sup>1</sup>

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## Abstract

The concept and the formalization of the arrival time in quantum mechanics are discussed. Different approaches based on trajectories, quantization rules, time operators, phase space techniques, renewal equations or operational procedures are reviewed or proposed. Open questions and loose ends are pointed out.

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<sup>1</sup>It is a pleasure to dedicate this article to Rolf Landauer in honor of his 70th birthday. His contributions and comments have stimulated our interest and motivated much of our work in “time in quantum mechanics”.

# 1 Introduction

In non-relativistic mechanics the system or its state can be characterized by a number of quantities with dimension of time. Some, such as a resonance lifetime, are state independent, and derived from the Hamiltonian; other characteristic times depend on the initial state and its dynamical evolution. Examples are the dwell, traversal [1], or arrival times, and generally the times required to find a dynamical variable with a given value [2]. The quantization of the classical characteristic times is problematic and has attracted considerable attention in recent years [3]. “Traversal” and “arrival” times have in particular been studied with many different formal treatments. We shall be mainly concerned with “arrival times” in this paper although the difficulties, techniques and ideas can be easily translated to other characteristic times.

## 1.1 Arrival times in classical mechanics

Assume a structureless particle that moves in one dimension with position  $q$  and momentum  $p$ . In simple cases the trajectory may cross a given spatial point  $X$  only once [4], but if a potential barrier reflects the trajectory, or if time dependent or noisy interactions affect the motion, the trajectory may cross  $X$  several times. The first crossing determines the *first passage time*, and in general, the  $n$ -th crossing gives the  $n$ -th *passage time*. For statistical ensembles of noninteracting particles there is a distribution of times associated with the  $n$ -th passage. Provided some restrictions are imposed on the initial ensemble, the average value and higher statistical moments can be defined from it. However, the distributions may exist even if the moments do not.

Let us examine first the free motion case. An ensemble of freely moving noninteracting particles will be described by a phase space distribution  $F(q, p, t)$ , normalized to one, that satisfies  $F(q, p \leq 0) = 0$ , i.e., all particles move rightwards. For free motion the trajectories cross the point  $X$  only once, and  $J(X, t)$ , the probability flux or current density at  $X$ , provides the distribution of first passage or arrival times;  $J(X, t)dt$  is the fraction of particles that cross  $X$  between  $t$  and  $t + dt$ . Using

$$J(X, t) = \int F(X, p, t) \frac{p}{m} dp, \quad (1)$$

(all integrals go from  $-\infty$  to  $\infty$  unless indicated otherwise. In this particular one the lower limit can of course be set to zero), and the trajectory equation,  $q(t) = q_0 + pt/m$ , the average arrival time is given for free motion by

$$\int J(X, t) t dt = \iint F(q_0, p, 0) \frac{(X - q_0)m}{p} dq_0 dp. \quad (2)$$

The integral exists if the singularity at  $p = 0$  is cancelled, at last partially, by  $F$ . An asymptotic behaviour  $F \sim p^\epsilon$ ,  $\epsilon > 0$ , when  $p \rightarrow 0$  is sufficient.

When the ensemble contains particles with momenta of arbitrary sign, or for more complex dynamics with multiple crossings,  $J$  is not the first passage time distribution any more. Suppose however that all particles are initially on the left side and that any particle that crosses  $X$  is eliminated at the crossing instant. When this “absorbing boundary” is imposed it is still true that the flux  $J_{ab}(X, t) = -dN/dt$  provides the (unnormalized) first passage time distribution. Here,  $N = N(t)$  is the time dependent (diminishing) “norm”  $N = \iint F(q, p, t) dq dp$ , and

$$J_{ab}(X, t) = \lim_{\epsilon \rightarrow 0} J(X - \epsilon, t), \quad \epsilon > 0. \quad (3)$$

(Equivalently, the particles that have crossed  $X$  at least once may be labeled, without affecting their dynamics, so that the first passage time distribution becomes proportional to the flux of the complementary subensemble of unlabeled particles.) If not all particles are eventually absorbed, the distribution is normalized by dividing  $J_{ab}(X, t)$  by the total norm absorbed,  $1 - N(\infty) = \int dt J_{ab}(X, t)$ .

A standard way to study first passage time distributions in classical stochastic diffusive processes is the use of “renewal equations” [5]. Assume that all members of the ensemble have initial position  $q_0 = x_0$  at  $t = 0$ . There are two ways to arrive at point  $x < X$ : Directly, via trajectories that remain in the left subspace (with respect to  $X$ ), or crossing  $X$  one or more times. Let  $P_0(x, x_0, t)$  be the probability density for the “direct” event. It can be obtained by solving the diffusion equation with absorbing boundary conditions at  $X$ . The probability of the “indirect” event can be decomposed according to the first passage time at  $X$  so that the probability density for being at  $x$  at time  $t$  irrespective of the path may be written as

$$P(x, x_0, t) = P_0(x, x_0, t) + \int_0^t f_{X, x_0}(t') P(x, X, t - t') dt'. \quad (4)$$

(A related equation was proposed by Schrödinger [6,7] in terms of cumulative probabilities.) This is the renewal equation, where  $f_{X, x_0}(t')$  is the first passage time distribution, and  $P(x, X, t - t')$  is the probability density for being at  $x$  at time  $t$  conditioned to having been at  $X$  at time  $t'$  [8].  $f_{X, x_0}(t')$  is the probability flux at  $X$  with an absorbing boundary, see (3), but it can also be obtained solving the integral equation by Laplace transform.

## 1.2 Quantum arrival times?

Due to the basic role of the trajectory concept to define “arrival” or “passage”, problems may be expected to translate the classical results to quantum

mechanics. Allcock examined the question some time ago and arrived at a negative conclusion [9], but more recently his arguments have been found to be too pessimistic by many researchers. In particular, his claim on the unavoidability of reflection by the detector has been shown to be unfounded [10,11]. The very existence of difficulties makes the time of arrival a fascinating subject where basic aspects of quantum mechanics have to be considered, such as the interpretation of the quantum formalism, the particle-wave dualism, ambiguities in the quantization of a classical variable, the relation between classical and quantum mechanics, or the “measurement problem”. These are all difficult and not completely understood matters so, not surprisingly, the arrival time, as well as any other problem concerning time in quantum mechanics is challenging and, frequently, controversial. Apart from the purely conceptual interest there are experiments that provide arrival times and a theoretical interpretation is required. Indeed, general first passage time problems (where the examined variable is not necessarily a position) are quite common. An example is the distribution of times required for a diatomic molecule to dissociate [5]. Several complementary research avenues will be discussed in the following sections. They do not necessarily conflict with each other, since different results are generally the consequence of having posed different questions [12]. Work on the various routes is at different levels of maturity and some of the many open questions are indicated. Because of length limitations this is not a truly exhaustive review, so we can barely touch the surface of many interesting contributions. We hope anyway that the uninitiated (and perhaps some experts) will benefit from the global perspective presented here.

## 2 Particle trajectories and wave features

Some interpretations or formulations of quantum mechanics are based on the trajectory concept. In the “causal” theory of Bohm [13], or in the stochastic interpretation [14], the trajectories are supposed to be actual ones and combine according to standard probability rules. (Instead, in Feynman’s path integral formalism the contribution of each path is added as a complex amplitude. Other formal paths have also been explored for certain applications, for example “Wigner trajectories”, based on a fit of the dynamical equation for the Wigner function into the classical Liouville equation, or “Weyl trajectories”, that retain several useful properties in common with the classical ones [15].) Using one of these theories definite, and generally different answers for characteristic times, such as the traversal time, the dwell time and the arrival time are found. The basic problem with these investigations is the uncertainty on the ultimate reality of the involved trajectories. Are, for example, Bohm trajectories anything more than a possible scenario? There is no experimental evidence to support or reject them as actual particle paths.

Finding a way to answer this question would be a major breakthrough in the foundations of quantum physics. In the “causal” approach to the arrival time [16],  $|J(X, t)|$  (suitably normalized) is the general arrival time distribution. The result holds for arbitrary  $X$  even in the presence of a potential barrier. Because all trajectory crossings, repeated or not, are taken into account,  $|J(X, t)|$  does not correspond in general to a *first passage* time distribution.

Instead of relying on the problematic “particle” aspect of the quantum state, it is also possible to pay attention to times characterizing the evolution of the “wave”, frequently by means of asymptotic methods. An example is the asymptotic “phase time” derived from a stationary phase argument. More detailed information is extracted from contour deformations in the complex plane and steepest descent methods [17]. Within this perspective, it is natural to define “times of arrival” of special wave features: the main peak, the forerunners, the centroid, or a given percentage of probability density. These may be very relevant when the detectors are sensitive to them, but should not be over-interpreted in terms of “particle paths”.

### 3 “Operational” procedures

The quantum arrival time may also be handled by means of experiments that would provide the traversal time classically. Quantum theory should be able to predict the statistics of the experimental observations using appropriate models for the measurement. (Inversely, it is also possible to derive from the statistics of the recorded data the associated quantum-mechanical observable [18].) These “operational methods” may provide bizarre results, possibly opposite to classical expectations. In fact different, classically equivalent experiments give in general different outcomes. This quantum multiplicity associated with a single classical quantity was already emphasized by Bohr [19]: *“Evidence obtained under different experimental conditions cannot be comprehended within a single picture, but must be regarded as complementary, in the sense that only the totality of the phenomena exhausts the possible information about the objects.”*

We shall describe first two operational approaches based on two implementations of an “absorbing boundary”.

#### 3.1 Elimination of norm at a discretized sequence of times

Assume that a wave packet is chopped sharply at a regular sequence of times  $\{t_i\}$ ,  $i = 1, 2, \dots$  separated by  $\Delta t$  in such a way that the part on the right of the point  $X$  is eliminated [20,21]. For a classical ensemble of noninteracting particles such a procedure would lead in the limit  $\Delta t \rightarrow 0$  to the first arrival

time distribution since the chopping is effectively acting as an absorbing boundary. However, in quantum mechanics the story is quite different. By increasing the chopping rate a higher fraction of particles is reflected, and in the limit  $\Delta t \rightarrow 0$  this process leads to total reflection.

### 3.2 Arrival detectors and complex potentials

The use of complex potentials has been proposed as a way to mimic the classical “absorbing boundary” procedure to evaluate first passage times, and to model experimental conditions in destructive time of flight experiments [11,22]. This approach does not lead to the disappointing result of the previous operational method. The processes taking place in the destructive detector, such as ionizations, change the structure or internal state of the incident particle, i.e., in scattering theory language, they change “the channel”. Of course the full measurement can be quite complex and involve many degrees of freedom, but a “reduced” Schrödinger equation can always be written for the incident channel in terms of a complex (non-hermitian) potential, whose precise form is usually modelled phenomenologically. The norm of the incident channel,  $N$ , is not conserved and may be used to define an effective arrival time distribution, proportional to  $-dN/dt$ , that will in general depend on the particular complex potential (equivalently on the detector). Under conditions usually met in time of flight experiments using atomic or molecular beams, - detector at asymptotic distance from the scattering region and particle source - and for good enough absorbers  $-dN/dt$  can be approximated by  $J$ , the quantum mechanical current density *without* the absorber. The difference between the first moments of  $-dN/dt$  and  $J$  (the time averages) is the *dwell time* of the original particle in the complex potential region [11]. In fact the “average”  $\int Jtdt / \int Jdt$  is an ideal, apparatus-independent quantity that can also be obtained in certain conditions as an average of a time operator (see the discussion below). Regarding the flux as the ideal arrival time distribution, for free motion, or at asymptotic distances from scatterers or sources, is an appealing idea because of the agreement with the classical expression. However, a quantum mechanical state composed by positive momenta is compatible with a negative value of  $J$  at certain times and positions [9,23]. Let us first stress that this effect is quantitatively negligible for normal practice [11], so that  $J$  should be a satisfactory quantity for the analysis of most experimental data. Nevertheless, the fundamental objection to regard  $J$  as a true arrival probability remains a valid one, and the deviations of experimental distributions from  $J$  in the quantum backflow regime can and should be studied with the aid of modern advances in experimental atomic and optical physics.

It would be interesting to determine if ideal absorbing conditions (reflection coefficients of the complex potential equal to zero for the wave packet momentum range) imply a unique potential, or any particular relation be-

tween  $-dN/dt$  and  $J$  (so that an ideal quantity could be defined from the operational procedure.) Inverse scattering for real potentials in one dimension is a well developed field but very little is known about the “inverse scattering problem” for complex potentials. A second open question is the explicit construction of models, with additional detector degrees of freedom, that justify the phenomenological results on a more fundamental level.

### 3.3 Other measurement models

Recently, Aharonov et al. [24] have analyzed a number of simple, idealized “toy models” of arrival time measurements. Unlike the previous approach, they explicitly include extra “detector” degrees of freedom in the (real) Hamiltonian. They conclude that the free particle arrival time cannot be measured more accurately than  $\hbar/E$ , where  $E$  is the kinetic energy of the particle, but it is necessary to investigate further if this is a model dependent result or a fundamental limitation (they argue in favor of the later).

Schulman has proposed a theory of quantum measurement where the state of the studied microscopic system evolves by unitary evolution -including generally environment and apparatus in the Hamiltonian- to one of the possible eigenstates of the measured observable (corresponding to the result of the measurement) [25]. In particular, the particle detection would require the localization of the entire particle wave function in the detector [26]. Thus this theory seemingly leads to different results from the ones discussed in the previous subsection, where such localization is not assumed. A detailed model including apparatus and environment would clarify the actual differences further, and the occurrence or not of the localization proposed; also the quantitative implications in the calculation of arrival times. Schulman has discussed experimental tests that would determine the validity of his theory [26,25].

## 4 Path decomposition expansion and renewal equations

A renewal equation with the form (4) or a related equation in terms of cumulative probabilities could be formally written in the quantum case and solved by Laplace transform by giving some precise meaning to the symbol  $P(x_2, x_1, t_2 - t_1)$  [7]. This meaning however is not at all obvious or clearly defined unless some interpretation in terms of trajectories is used where the position becomes a Markov process. Localizing the particle around  $x_1$ , e.g. with a Gaussian wave function, is possible [7], but the evolution will depend on the momentum average and dispersion of the chosen Gaussian.

A close quantum relative of the classical renewal equation is the “path decomposition expansion” (PDE) of the propagator into a sum over Feynman paths classified according to their first passage time at  $X$ . For  $x, x'' < X$  it takes the form

$$K(x'', t''|x, 0) = K_0(x'', t''|x, 0) + \int_0^{t''} K(x'', t''|X, t') \frac{i\hbar}{2m} \frac{d}{dX} K_0(X, t'|x, 0) dt', \quad (5)$$

where  $K_0$  is a restricted propagator corresponding to the half-space  $(-\infty, X)$ . This expression was first derived using Feynman path integrals [27], and later by a more general operator procedure [28]. An even simpler derivation follows from the general relation between the propagators  $K$  and  $K_0$ , corresponding to Hamiltonians  $H$  and  $H_0$ ,  $H = H_0 + V$ ,

$$K(t) = K_0(t) - \frac{i}{\hbar} \int_0^t K(t-t') V K_0(t') dt', \quad (6)$$

by putting  $H_0 = \Theta(-x+X)H\Theta(-x+X)$  ( $\Theta$  is the Heaviside function) [29]. By analogy with the renewal equation, it is tempting to consider

$$A \equiv \int \frac{i\hbar}{2m} \frac{d}{dX} K_0(X, t'|x, 0) \psi(x, 0) dx \quad (7)$$

as a “first passage time amplitude” [28,30]. Some caution should however be exercised since  $K_0$  does not correspond to an absorbing boundary but to a reflecting one, so the analogy with (4) is only a partial one. Moreover, the squared modulus does not have the correct dimensions nor will generally satisfy “probability sum rules” because of interferences between paths taking different times. The interferences may however disappear when coupling the particle with an environment. This decoherent effect and the probabilities so obtained have been examined by Halliwell and Zafiris [31] within the “decoherent histories approach to quantum mechanics”. These authors point out that when decoherence is achieved the resultant probabilities depend on the mechanism producing decoherence, and insist, quoting Landauer [1,32], that time in quantum mechanics only makes sense if the mechanism by which it is measured is fully specified.

It is illustrative to compare the (appropriately normalized) squared modulus of  $A$  with the flux  $J$  [29]. For free motion,  $K$  and the restricted propagator  $K_0$  are known, and analytical results are available by taking as initial state, at time  $t = 0$ , a minimum uncertainty Gaussian wave function with central position and momentum  $x_0$  and  $p_0$ , and spatial variance  $\delta^2$ . If  $X = 0$ ,

$$J(0, t) = \left(\frac{2}{\pi}\right)^{1/2} \frac{(4\delta^4 p_0 m - \alpha) m \delta}{[(t\hbar)^2 + (2\delta^2 m)^2]^{3/2}} e^{-\frac{2\delta^2 [x_0^2 m^2 + 2m p_0 t x_0 + p_0^2 t^2]}{(t\hbar)^2 + (2\delta^2 m)^2}}, \quad (8)$$



where  $\alpha = tx_0\hbar^2$ . A detailed calculation shows that, putting  $\alpha = 0$ , the right hand side is proportional, up to time independent factors, to  $|A|^2$ . The shapes of  $J$  and a (normalized)  $|A|^2$  are close to each other when  $2\delta^2 p_0 \gg |x_0|\hbar$ .

## 5 Quantization rules and time operators

The search for “time operators” and the study of their properties has been the traditional and most popular approach [2,9,33-47], even though Pauli pointed out the impossibility of a self-adjoint time operator conjugate to a Hamiltonian with bounded spectrum [48]. There are however different ways to circumvent this objection by defining operators which retain at least partially the desirable properties of a time observable. Unfortunately, the definition of quantum operators associated with classical quantities is not justified at present by any fundamental quantization theory, and all known quantization rules are essentially heuristic recipes that may provide ambiguous, non unique, or useless operators for some classical quantities [49]. It is essential in each case to examine the properties of the operators obtained and determine their physical content (conditions that they satisfy, domain of applicability, and relation to operational procedures and other quantities of interest). The connection of the operators with actual measurements is frequently obscure: As stated by Wigner [50], “There is no rule that would tell us which self-adjoint operators are truly observables, nor is there any prescription known how the measurements are to be carried out, what apparatus to use, etc. In a theory with a positivistic undertone, this is a serious gap.” In principle, any operator resulting from a quantization rule can be associated with a property of the state of the system, which may or may not be easily measurable or useful. In general, a number of conditions, not only motivated by experiments, are imposed to select among the possible operators. Claims of uniqueness should then be taken cautiously, since they generally reflect the proponent bias towards a group of conditions, which may not be satisfactory for certain purposes (and certainly not for the sensibility of competitors!)

Kijowski [36], for the free motion case, imposing a series of conditions compatible with the classical arrival time, and limiting the domain to states with positive momentum, derived (uniquely within the stated conditions) the distribution of arrival times ( $X = 0$ )

$$\Pi(t; \psi(0)) = \left| \frac{1}{(m\hbar)^{1/2}} \int_0^\infty \sqrt{p} e^{-ip^2 t/2m\hbar} \langle p | \psi(0) \rangle dp \right|^2, \quad (9)$$

where  $\langle p | \psi(0) \rangle$  is the state in momentum representation at  $t = 0$ . Note the correct behaviour under time translation of the wave function:  $\Pi[t; \psi(0)] =$

$\Pi[t - t'; \psi(t')]$ . Related analysis were carried out by Werner [37] and Ludwig [38], and more recently by several authors [2,39-43]. This distribution has also the remarkable property of having the same first moment than the flux [36],

$$\int \Pi(t) t dt = \int J(t) t dt, \quad (10)$$

and may be formally written as the square  $|\langle t|\psi\rangle|^2$ , where  $|t\rangle$  is the eigenstate,

$$\langle p|t\rangle = \left(\frac{p}{m\hbar}\right)^{1/2} e^{ip^2 t/2m\hbar}, \quad (11)$$

of the time operator [44,46,41]

$$\hat{t} = -\frac{m}{2} \left( \hat{q} \frac{1}{\hat{p}} + \frac{1}{\hat{p}} \hat{q} \right) \quad (12)$$

(The symmetrical form  $-m\hat{p}^{-1/2}\hat{x}\hat{p}^{-1/2}$  leads in the positive momentum subspace to the *same* eigenvalue equation and eigenstates as  $\hat{t}$  [2,39].) The normalization in (11) is chosen so that

$$\int \langle p|t\rangle \langle t|p'\rangle dt = \delta(p - p'), \quad p, p' > 0. \quad (13)$$

Dealing with states composed by coherent combinations of momenta with arbitrary sign is more difficult because of the singularity at momentum zero, but there should be a theoretical distribution in agreement with an experimental arrival time distribution even if the average does not exist. A regularization procedure has been proposed [39]. The results can be very different in this case from the ones derived from the Bohm approach [51], and consideration of actual experiments in this regime would be of much interest.

## 5.1 Phase space quantization techniques

Quantum states and “observables” can be expressed equivalently in operator form ( $\hat{\rho}$  and  $\hat{G}$  respectively) or by means of various phase space representations or images [ $F(q, p)$  and  $g(q, p)$  respectively] in such a way that the expectation value of the operator can be written as

$$\langle \hat{G}(\hat{q}, \hat{p}) \rangle = \iint F(q, p) g(q, p) dq dp. \quad (14)$$

Sets of four transformations

$$\hat{\rho} \xleftrightarrow{\quad} F, \quad \hat{G} \xleftrightarrow{\quad} g, \quad (15)$$

characterized by a kernel function  $f$  can be constructed [52-53]. Each  $f$  defines the quantization rule  $g \rightarrow \hat{G}$ ,

$$\hat{G}(\hat{q}, \hat{p}) = \frac{1}{4\pi^2} \iiint g(q, p) e^{-i(\theta q + \tau p)} f(\theta, \tau) e^{i(\theta \hat{q} + \tau \hat{p})} dq dp d\theta d\tau, \quad (16)$$

and the phase space images of states and observables. The Weyl-Wigner formalism, where the state is represented by the Wigner function,  $F^W$ , and the quantization rule is given by Weyl's prescription, corresponds to  $f = 1$  [54].

We shall first look at the free motion case. The classical time of arrival at point  $X$  when the trajectory starts at  $q_0, p$  at time  $t = 0$  is given by

$$t = \frac{(X - q_0)m}{p}. \quad (17)$$

Inserting (17) in (16) the corresponding operators are obtained as

$$\begin{aligned} \hat{t} &= \frac{Xm}{2\pi} \iint \frac{1}{p} e^{-i\tau(p-\hat{p})} f(0, \tau) dp d\tau \\ &- \frac{m}{2\pi i} \iint \frac{1}{p} \left[ f'_{\theta=0}(\theta, \tau) + f(0, \tau) \left( \frac{1}{2} i\hbar\tau + i\hat{q} \right) \right] e^{-i\tau(p-\hat{p})} dp d\tau. \end{aligned} \quad (18)$$

A wide family of quantization rules, and in particular the ones by Weyl, Rivier and Born-Jordan [52,53] lead to the same operator we have already discussed, expressed now for arbitrary  $X$ ,

$$\hat{t} = \frac{Xm}{\hat{p}} - m \left( \hat{q} \frac{1}{\hat{p}} - \frac{\hbar}{2i} \frac{1}{\hat{p}^2} \right) = \frac{Xm}{\hat{p}} - \frac{m}{2} \left( \hat{q} \frac{1}{\hat{p}} + \frac{1}{\hat{p}} \hat{q} \right), \quad (19)$$

while the standard and antistandard quantizations do not produce a hermitian operator so they will not be discussed further. Several properties of this operator are easily proved in phase space using the Weyl-Wigner formalism,  $f = 1$ . In particular, the relation (10) can be derived using  $F^W(q, p, t) = F^W(q_0, p, 0)$  (valid for free motion,  $q = q_0 + tp/m$ ), and noting that the phase space representative of the flux operator is the classical expression  $\delta(X)p/m$ .

## 6 Other phase space techniques

Generalizing the free motion arrival time operator is not simple, and only a few interaction potentials [46,47] or asymptotic distances from a scattering

potential [42] have been worked out. The classical expression for the trajectory may become very involved, and it is rarely explicit (as a function of  $q_0$ ,  $p_0$ , and  $t$ ), so that considering different operator orderings becomes cumbersome or impossible in practice. There is the additional difficulty that the equation  $X = q(q_0, p_0, t)$  has in general real solution only for a limited domain of the initial phase space.

We shall sketch here how the phase space formalism combined with the Heisenberg picture can be used to provide such a generalization, and to deal with quantities different from position. The phase space images  $g^H$  of the Heisenberg operators  $\hat{G}(t)$  depend on the initial phase space point  $(q_0, p_0)$  and on time,

$$\text{tr} [\hat{\rho}(0)\hat{G}(t)] = \iint F(q_0, p_0, 0)g^H(q_0, p_0, t)dq_0dp_0. \quad (20)$$

(In particular the images of the Heisenberg operator for position,  $x^H(q_0, p_0, t)$ , are in the classical limit classical trajectories.) Suppose that the equation for an arbitrary  $g^H$ ,

$$g^H(q_0, p_0, t) = \mathcal{G}, \quad (21)$$

where  $\mathcal{G}$  is a predetermined value, has at least one solution for  $t > 0$ . We can identify the sequence of times  $t_f^{(i)}(q_0, p_0, \mathcal{G})$ ,  $i = 1, 2, \dots$  where  $g^H(q_0, p_0, t)$  “crosses”  $\mathcal{G}$ . An average time (note the dependence on  $f$ ) is then defined for the  $i$ -th crossing as

$$\langle t_i \rangle = \frac{1}{N_i} \iint_{D_i} F(q_0, p_0, 0)t_f^{(i)}(q_0, p_0, \mathcal{G})dq_0dp_0, \quad (22)$$

$$N_i = \iint_{D_i} F(q_0, p_0, 0)dq_0dp_0, \quad (23)$$

where the domain of integration is restricted to the phase space region  $D_i$  where (21) has an  $i$ -th solution,  $N_i$  is a normalization constant, and  $f$  may be tailored in order to satisfy consistency requirements [36] or experimentally obtained values. Higher moments can be obtained similarly.

## 7 Average “presence” times

In several of the previous sections the probability flux  $J$  has been emphasized as an important quantity in relation to the arrival time. We have seen in particular that classically it provides the first passage distribution for free motion, or for a general case when absorbing boundaries are imposed. The probability density  $\rho$  does not play this role. However, average times can

also be defined in terms of it. It is appropriate to use a different name for them, for example average “presence” times,

$$\tilde{t} = \frac{\int \rho(X, t) t dt}{\int \rho(X, t) dt} \quad (24)$$

They are of interest for detectors sensitive to the presence of the particle rather than to the flux. These times, and associated operators have been studied by several authors [55-58].

## 8 Concluding remarks

Understanding the various aspects of the time of arrival in quantum mechanics remains an exciting technical and conceptual challenge. Several theoretical approaches have been proposed or reviewed. Experiments in non-classical regimes (with backflow, or for motion governed by an interaction potential) would provide a much needed reference to refine operational models and ascertain the practical relevance of intrinsic quantities or operators.

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